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On Pólya–Friedman random walks

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Abstract
The Pólya process is an urn scheme arising in the context of contagion spreading. It exhibits unstable persistence effects. The Friedman urn process is dual to the Pólya one with antipersistent stabilizing effects. It appears in a safety campaign problem. A Pólya–Friedman urn process is investigated with a tuning persistence parameter extrapolating the latter two extreme processes. The study includes the diffusion approximations of both the Pólya–Friedman proportion process and the population gap random walk. The structure of the former is a generalized Wright–Fisher diffusion appearing in population genetics. The correlation structure of the latter presents an anomalous character at a critical value of the persistence parameter.

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1. Introduction and outline of the results

The Pólya process is a discrete-time urn scheme arising in the context of contagion or rumor spreading. It exhibits unstable inflationary effects. The Friedman urn process is dual to that of Pólya with antipersistent deflationary effects. It appears in a safety campaign problem. Both models belong to the class of time-dependent Markoff chains, a key property leading to specific probabilistic issues.

In this paper, a Pólya–Friedman urn process is investigated with a tuning persistence parameter ρ, $|\rho| \leqslant 1$, extrapolating the latter two extreme processes (corresponding to $\rho = 1$ and $\rho = -1$, respectively).

In section 2, we first revisit the celebrated Pólya process; details include the urn composition statistics and a short introduction to the discrete-time proportion process which plays an important role in the exchangeability property of the increments sequence. Then, we investigate a continuous spacetime diffusion approximation of the discrete proportion process. We show that it leads to a time-inhomogeneous Wright–Fisher diffusion process, a basic model arising in neutral mathematical population genetics [4]. We compare the discrete and scaling limit processes by means of an adequate time substitution. In section 3, we focus on the population gap process measuring the excess population of the two competing types of individuals present in the population at all times. We obtain a diffusion approximation of a scaled version of this process. By doing so, we obtain some insight into the limiting Gaussian behavior of this quantity.

In section 4, we start by listing some statistical properties of the Friedman urn model before concentrating on the full Pólya–Friedman urn process, tuned by ρ. The study includes the diffusion approximations of the Pólya–Friedman proportion process and of its associated population gap random walk; it follows the path used in the previous study for the 'particular' Pólya process. Using similar tools, we show that the structure of the scaled Pólya–Friedman proportion process is a generalized Wright–Fisher diffusion about which very little seems to be known. We also show that both the variance and the autocorrelation structure of the Pólya–Friedman population gap random walk present a qualitative change at the critical value of the persistence parameter $\rho = 1/2$. This fact is reminiscent of a dynamical phase transition. The study makes use of a linear Gaussian time-inhomogeneous diffusion approximation for this process.

2. The Pólya urn model

We start with generalities on a model attributed to Pólya.

2.1. The discrete Pólya model: a reminder

Interest in Pólya urn processes stems from the following contagion process: an initial population constituted of two types of genes is randomly pairing, one at a time, with an unlimited external reservoir of neutral individuals while transmitting to them their genes in the contact process. The types could also be the religious or political beliefs of a group of initiates randomly sampled in a propaganda campaign to switch to their cause a large set of initially neutral people in spreading of the rumor process. Then, the current and limiting population sizes of each type deserve interest.

The mathematical formulation of this model is in terms of an urn problem [7, 12]: an urn initially contains N balls, n_1 of which are of type 1, $n_0 \equiv N - n_1$ of type 0. The composition of the urn evolves in time. We shall let N_n^1 be the number of type 1 balls within the urn at time n, starting from $n_1 = [Nx_1]$ type 1 balls, where $x_1 \in (0, 1)$. (Possibly, we may wish to consider the case of a random unknown initial state N_0^1 within the urn, in which case we have to randomize n_1 or x_1.)

The evolution mechanism is that of the contamination model of Pólya: a ball is drawn and then replaced in the urn, along with a ball of the same color drawn so that at time n there are $N + n$ balls within the urn, N_n^1 of which being type 1.

Let $P_n \equiv N_n^1 \big/ (N + n)$ be the fraction (or proportion) of type 1 balls at time n and $(U_n; n \geqslant 1)$ an independent and identically distributed (iid) sequence, uniformly distributed on $[0, 1]$. The discrete dynamics of the Pólya process is described by

$$N_{n+1}^1 = N_n^1 + \mathbf{1}_{U_{n+1} \leqslant P_n},$$

hence, given $N_n^1 = k \in \{n_1, \dots, n_1 + n\}$

$$N_n^1 = k \to k + 1 \text{ with probability } p_{k,n} = \frac{k}{N + n} = \frac{1}{2}\left(1 + \frac{2k - (N + n)}{N + n}\right) \tag{1}$$

$$N_n^1 = k \to k \text{ with probability } q_{k,n} = 1 - \frac{k}{N + n} = \frac{1}{2}\left(1 - \frac{2k - (N + n)}{N + n}\right). \tag{2}$$

The striking feature of these transition probabilities is that they depend on time n. This model is in the class of Markoff chains inhomogeneous in time: the origin of time-inhomogeneity is that there is no conservation of the number of balls within this unbalanced urn process. Note that, except maybe for the initial condition, the dynamics of $N_n^0 = (N+n) - N_n^1$ is the same as that of N_n^1 in that given $N_n^0 = k \in \{n_0, \dots, n_0 + n\}$

$$N_n^0 = k \to k+1 \text{ with probability } p_{k,n}$$
$$N_n^0 = k \to k \text{ with probability } q_{k,n}.$$

The dynamics of N_n^1 obeys the classical transitory Pólya distribution (see [6], proposition 3):

$$\mathbb{P}\left(N_n^1 = k\right) = \frac{\binom{k-1}{n_1-1}\binom{N+n-k-1}{N-n_1-1}}{\binom{N+n-1}{N-1}}, \qquad k \in \{n_1, \dots, n_1+n\}. \tag{3}$$

Let $M_n \equiv N_n^1 - N_{n-1}^1$ and recall $N_0^1 = n_1$. Then, $(M_1, \dots, M_n) \in \{0,1\}^n$ may be viewed as a sequence of binary digits labeling the successive n-draws or jumps, and it makes sense to compute the probability of a particular n-string. We have

$$\mathbb{P}(M_1 = m_1) = m_1 \frac{n_1}{N} + (1-m_1)\left(1 - \frac{n_1}{N}\right),$$

and for $k \geqslant 2$: $\mathbb{P}(M_k = m_k \mid M_1, \dots, M_{k-1})$

$$= m_k \left(\frac{n_1 + \sum_{l=1}^{k-1} \mathbf{1}(M_l = m_k)}{N+k-1}\right) + (1-m_k)\left(1 - \frac{n_1 + \sum_{l=1}^{k-1} \mathbf{1}(M_l = m_k)}{N+k-1}\right).$$

Proceeding in this way, the joint distribution of $(M_1, \dots, M_n)$ reads

$$\mathbb{P}(M_1 = m_1, \dots, M_n = m_n)$$
$$= \prod_{k=1}^{n} \left\{ m_k \frac{n_1 + \sum_{l=1}^{k-1} \mathbf{1}(M_l = m_k)}{N+k-1} + (1-m_k)\left(\frac{n_0 + \sum_{l=1}^{k-1} \mathbf{1}(M_l = 1 - m_k)}{N+k-1}\right)\right\}$$
$$= \frac{(n_1)_{k_1}(n_0)_{k_0}}{(N)_n},$$

where $(n)_k \equiv n(n+1), \dots, (n+k-1)$ and $k_m \equiv \sum_{k=1}^{n} \mathbf{1}(m_k = m)$ is the number of occurrences of symbol (type or jump) $m \in \{0,1\}$ in the n-string, satisfying $k_0 + k_1 = n$. This distribution only depends on k_0 and k_1 and not on the particular ordering of the symbols in the chain. In other words, it is invariant under the permutations of the entries in that for all permutation σ of $\{1, \dots, n\}$:

$$\mathbb{P}(M_k = m_k; k \in \{1, \dots, n\}) = \mathbb{P}(M_k = m_{\sigma(k)}; k \in \{1, \dots, n\}).$$

The sequence $(M_1, \dots, M_n)$ is called exchangeable, after de Finetti. Thus, all possible sequences presenting k_0 (respectively k_1) occurrences of symbol $\{0\}$ (respectively $\{1\}$) in the n-string are equiprobable and there are $\binom{n}{k_0}$ such sequences: the probability to get a n-string with k_m occurrences of symbols $m \in \{0,1\}$ therefore is

$$\mathbb{P}\left(\sum_{k=1}^{n} M_k = k_1\right) = \frac{\binom{n_1+k_1-1}{k_1}\binom{n_0-k_0-1}{k_0}}{\binom{N+n-1}{n}} \tag{4}$$

which, as required, is also $\mathbb{P}\left(N_n^1 = n_1 + k_1\right)$ of (3) because $N_n^1 = n_1 + \sum_{k=1}^{n} M_k$. The latter distribution is known as the Dirichlet binomial distribution.

The sequence $(M_1, \dots, M_n)$ is called a Pólya urn sequence, enjoying the distinguished exchangeability property. Finally, we observe that

$$\mathbb{P}(M_1 = m_1, \dots, M_n = m_n) = \int_0^1 x^{k_1}(1-x)^{k_0} u(x)\,\mathrm{d}x$$

where $u(x) = \frac{\Gamma(N)}{\Gamma(n_0)\Gamma(n_1)} x^{n_1-1}(1-x)^{n_0-1}$ is the density of a beta$(n_1; n_0)$ distributed random variable on [0, 1]. Moreover,

$$\mathbb{P}\left(\sum_{k=1}^{n} M_k = k_1\right) = \binom{n}{k_0} \int_0^1 x^{k_1}(1-x)^{k_0} u(x)\,\mathrm{d}x \tag{5}$$

and the Dirichlet binomial distribution may be viewed as a randomized binomial distribution with mixing beta density u. The interpretation of this probability density arises from the observation that $\frac{1}{n}\sum_{k=1}^{n} M_k$ converges in distribution to a random variable P with density u. Although not iid, the increments $(M_n; n \geqslant 0)$ are conditionally iid Bernoulli given P, being 1 with probability P and 0 with probability $1-P$.

A simple way to observe this is as follows: the normalized proportion process $P_n = N_n^1 \big/ (N+n)$ is a martingale with values in [0, 1] converging almost surely (a.s.) to a non-degenerate limit P. Indeed

$$\mathbb{E}(P_{n+1} \mid P_n = x) = \frac{x(N+n)+1}{N+n+1} p_{x(N+n),n} + \frac{x(N+n)}{N+n+1} q_{x(N+n),n} = x.$$

Moreover, as it can be checked from the distribution of N_n^1 displayed in (3), that

$$P_n \equiv N_n^1 \big/ (N+n) \xrightarrow{d}_{n\uparrow\infty} P$$

with law beta$(n_1; N-n_1)$:

$$\mathbb{P}(P \in \mathrm{d}x) = \frac{\Gamma(N)}{\Gamma(n_1)\Gamma(N-n_1)} x^{n_1-1}(1-x)^{N-n_1-1}\,\mathrm{d}x \equiv u(x)\,\mathrm{d}x. \tag{6}$$

Now, $P_n = \left(n_1 + \sum_{k=1}^{n} M_k\right) \big/ (N+n)$ and so both P_n and $\frac{1}{n}\sum_{k=1}^{n} M_k$ converge to the same limit as $n \to \infty$. This result dates back to Pólya (1931), [12].

When N is large, if $n_1 = [Nx_1]$, the mean value of P is $[Nx_1]/N \sim x_1$. The order-2 moment is (if $\varepsilon \equiv 1/N$):

$$\frac{[Nx_1]([Nx_1]+1)}{N(N+1)} \sim x_1(x_1 + \varepsilon(1-x_1)),$$

so that the variance of P is of order $\varepsilon x_1(1-x_1) \to 0$.

The limit P is random and so P_n is sensitive to the initial conditions, translating the fact that the distribution of the Pólya urn composition is widely spread. When N is large, by the Stirling formula

$$\frac{\Gamma(N)}{\Gamma([Nx_1])\Gamma(N-[Nx_1])} \sim \frac{1}{\sqrt{2\pi N x_1(1-x_1)}} \left(x_1^{x_1}(1-x_1)^{1-x_1}\right)^{-N}$$

and

$$u(x) \sim \frac{1}{\sqrt{2\pi N x_1(1-x_1)}} \left(\left(\frac{x}{x_1}\right)^{x_1} \left(\frac{1-x}{1-x_1}\right)^{1-x_1}\right)^{N},$$

which is very peaked around x_1. P admits a variance that goes to 0 and $P \to \delta_{x_1}$: if N is large, one recovers asymptotically the initial proportion of type 1 balls, which may be useful if the initial composition is fixed but unknown.

For the population ratio of type 1 to 0 balls

$$\frac{N_n^1}{N_n^0} = \frac{P_n}{1-P_n} \to_{n\uparrow\infty} W = \frac{P}{1-P} > 0 \text{ a.s.}$$

where the random variable W admits a generalized Pareto density $v(z)$, $z > 0$ given by

$$v(z) = \frac{\Gamma(N)}{\Gamma(n_1)\Gamma(n_0)} z^{n_1-1}(1+z)^{-N} \underset{z\uparrow\infty}{\sim} \frac{\Gamma(N)}{\Gamma(n_1)\Gamma(n_0)} z^{-(n_0+1)}, \tag{7}$$

a power law with exponent n_0.

2.2. Diffusion approximation of the Pólya urn proportion process: the Wright–Fisher model

When N is large, we can rather work on the novel rescaled [0, 1]-valued random process

$$X_t = N^1_{[tN]}\big/(N + [tN]) = P_{[tN]}, \tag{8}$$

in continuous time t. Here, a time unit of X_t corresponds to a lapse of time N for the discrete original process N^1_n. The dynamics of X_t can be approximated by an inhomogeneous Itô diffusion equation with 'small' noise which should capture the essential features of N^1_n. We shall briefly sketch how this works and show with an example the discrepancies between the discrete and continuous (spacetime) processes. To be more precise, we may state:

When N gets large (or $\varepsilon \equiv N^{-1}$ gets small), the rescaled Pólya urn proportion process, namely, $X_t = N^1_{[tN]}\big/(N + [tN])$, obeys the neutral Wright–Fisher stochastic differential equation (SDE) in the sense of Itô:

$$\mathrm{d}X_t = \frac{\sqrt{\varepsilon X_t(1 - X_t)}}{1 + t}\,\mathrm{d}B_t, \qquad X_0 = x_1 \in (0, 1). \tag{9}$$

Here, B_t is the standard Brownian motion and so X_t is a martingale with no drift: $f(x) \equiv 0$ and with time-dependent local standard deviation given by $g_{\varepsilon,t}(x) \equiv \frac{\sqrt{\varepsilon x(1-x)}}{1+t}$.

We shall now give a sketch of proof. Formally, with $\delta X_t = X_{t+\varepsilon} - X_t$ and $(\zeta_t;\, t \in \varepsilon\mathbb{Z})$ an iid standard Gaussian sequence, we look for the drift f and the local standard deviation $g_{\varepsilon,t}$ such that

$$\delta X_t = f(X_t)\varepsilon + \sqrt{\varepsilon}\, g_{\varepsilon,t}(X_t)\zeta_{t+\varepsilon}.$$

From (8), we have

$$\begin{aligned}\delta X_t &= \frac{N^1_{[(t+\varepsilon)N]}}{N + [(t+\varepsilon)N]} - \frac{N^1_{[tN]}}{N + [tN]} = \frac{N^1_{[tN]+1}}{N + [tN] + 1} - \frac{N^1_{[tN]}}{N + [tN]}\\ &= \frac{N^1_{[tN]} + \delta N^1_{[tN]}}{N + [tN] + 1} - \frac{N^1_{[tN]}}{N + [tN]} = \frac{(N + [tN])X_t + \delta N^1_{[tN]}}{N + [tN] + 1} - X_t\\ &= \frac{\delta N^1_{[tN]} - X_t}{N + [tN] + 1}.\end{aligned}$$

Given $X_t = x$, the law of $\delta N^1_{[tN]}$ is

1 with probability x

0 with probability $1 - x$.

Thus,

$$\begin{aligned}f(x)\varepsilon \equiv \mathbb{E}(\delta X_t) &= \frac{(1 - x)}{N + [tN] + 1}x - \frac{x}{N + [tN] + 1}(1 - x) = 0\\ \varepsilon g^2_\varepsilon(x) \equiv \sigma^2(\delta X_t) &= \frac{(1 - x)^2}{(N + [tN] + 1)^2}x + \frac{x^2}{(N + [tN] + 1)^2}(1 - x)\\ &= \frac{x(1 - x)}{(N + [tN] + 1)^2}\end{aligned}$$

leading to a null drift: $f(x) = 0$ and to the local standard deviation

$$g_{\varepsilon,t}(x) = \frac{\sqrt{x(1 - x)}}{\sqrt{\varepsilon}(\varepsilon^{-1} + [t/\varepsilon] + 1)} = \frac{\sqrt{\varepsilon x(1 - x)}}{1 + \varepsilon([t/\varepsilon] + 1)} \sim \frac{\sqrt{\varepsilon x(1 - x)}}{1 + t}.$$

Thus, we obtain a diffusion process inhomogeneous in time whose local variance depends on time. This diffusion process has the time-dependent Fokker–Planck–Kolmogorov backward (respectively forward) infinitesimal generator:

$$G_t(\cdot) = \frac{\varepsilon}{2(1 + t)^2}x(1 - x)\partial^2_x(\cdot) \left(\text{respectively, } G^*_t(\cdot) = \frac{\varepsilon}{2(1 + t)^2}\partial^2_x[x(1 - x)\cdot]\right).$$

2.3. Continuous-time Pólya: the time substitution

Let

$$\tau_t = \int_0^t (1+s)^{-2}\,\mathrm{d}s = \frac{t}{1+t} \tag{10}$$

be an increasing time change. If $t_\tau = \frac{\tau}{1-\tau}$ is the inverse of τ_t, the new process $\mathcal{X}_\tau \equiv X_{t_\tau}$ now is a time-homogeneous Brownian motion with local variance $\varepsilon x(1-x)$. Indeed, the forward (respectively backward) infinitesimal generator of the time-changed process $\mathcal{X}_\tau$ becomes: $\mathcal{G} = (1+t)^2 G_t = \frac{1}{2}\varepsilon x(1-x)\partial_x^2$ (respectively: $\mathcal{G}^* = (1+t)^2 G_t^* = \varepsilon\partial_x^2[x(1-x)\cdot]$).

The underlying process is the celebrated Wright–Fisher model of population genetics, now obeying the time-homogeneous SDE:

$$\mathrm{d}\mathcal{X}_\tau = \sqrt{\varepsilon\mathcal{X}_\tau\,(1-\mathcal{X}_\tau)}\,\mathrm{d}B_\tau, \qquad X_0 = x_1 \in (0,1). \tag{11}$$

This SDE is also the continuous spacetime limit of the Moran model when a population of size N evolves in discrete time by choosing at random a pair of individuals, one of which will duplicate, the other will die out while preserving the total number N of individuals at each generation (the constant-size branching process) (see [1, 4, 10, 11]).

When looking for the long-time ($t \to \infty$) behavior of X_t, we need to compute the transitory solution of the neutral time-homogeneous Wright–Fisher model $\mathcal{X}_\tau$ till time $\tau = 1$. It turns out that the generators of the neutral Wright model admit a discrete spectrum with known eigenfunctions (Bernoulli polynomials) and known eigenvalues. Therefore, the current distribution of $\mathcal{X}_\tau$ is accessible, allowing to compute the limiting moments and the limit law of X_t ($t \to \infty$) simply by considering the moments and law of $\mathcal{X}_\tau$ till time $\tau = 1$. We shall briefly sketch how this works (see [10, 11] for additional details).

2.4. Limiting moments of X_t

The Bernoulli polynomials $B_k(x), k \geqslant 1$ are defined by

$$t\,\frac{\mathrm{e}^{xt}-1}{\mathrm{e}^t-1} = \sum_{k\geqslant 1} \frac{B_k(x)}{k!} t^k.$$

Let $u_k(x) = (-1)^{k-1}k!B_k(x)$. Then, $u_k(x), k \geqslant 1$ are the eigenfunctions of the infinitesimal generator $\mathcal{G} = \frac{1}{2}\varepsilon x(1-x)\partial_x^2$ associated with the eigenvalues $\lambda_k = -\varepsilon k(k-1)/2, k \geqslant 1$:

$$\mathcal{G}(u_k(x)) = \lambda_k u_k(x).$$

For instance, $u_1(x) = x$, $u_2(x) = x - x^2$, $u_3(x) = x - 3x^2 + 2x^3$, $u_4(x) = x - 6x^2 + 10x^3 - 5x^4 \ldots$.

The system $u_k(x), k \geqslant 2$ constitutes a complete orthogonal system of eigenvectors. Thus, for all function $\varphi(x)$ admitting a decomposition within the basis $u_k(x)$

$$\mathbb{E}^{x_1}\varphi(X_\infty) = \mathbb{E}^{x_1}\varphi(\mathcal{X}_\tau) = \sum_{k\geqslant 2} c_k\,\mathrm{e}^{\lambda_k\tau} u_k(x_1) \tag{12}$$

where $\langle\varphi, u_k\rangle_w \equiv \int_0^1 \varphi(x)u_k(x)w(x)\,\mathrm{d}x$, $w(x) = 1/(x(1-x))$ and $c_k \equiv \frac{\langle\varphi,u_k\rangle_w}{\langle v_k,u_k\rangle}$.

In particular, with $x^k = \sum_{l=1}^k c_{k,l}u_l(x)$ where the coefficients $c_{k,l} \in \mathbb{Z}$:

$$\mathbb{E}^{x_1}\left(X_\infty^k\right) = \mathbb{E}^{x_1}\left(\mathcal{X}_{\tau=1}^k\right) = \sum_{l=1}^k c_{k,l}\,\mathrm{e}^{\lambda_l} u_l(x_1)$$

are the limiting moments of $X_t, t \to \infty$.

For the first two moments, we obtain

- $\mathbb{E}^{x_1}(X_\infty) = \mathbb{E}^{x_1}(\mathcal{X}_1) = u_1(x_1)\mathrm{e}^{-0} = x_1$.
- $\mathbb{E}^{x_1}\left(X_\infty^2\right) = \mathbb{E}^{x_1}\left(\mathcal{X}_1^2\right) = u_1(x_1)\,\mathrm{e}^{-0} - u_2(x_1)\,\mathrm{e}^{-\varepsilon} = x_1(1-(1-x_1)\,\mathrm{e}^{-\varepsilon}) \sim_{\varepsilon\downarrow 0} x_1(x_1+\varepsilon(1-x_1))$ so that the limiting variance of X_t is: $\sigma_{x_1}^2(X_\infty) \sim \varepsilon x_1(1-x_1)$.

As required, the laws of P as from (6) and of X_∞ admit comparable first and second order moments in the small ε limit.

2.5. *Limit law of X_t*

The eigenfunctions of the dual Kolmogorov operator $\mathcal{G}^*(\cdot) = \frac{\varepsilon}{2}\partial_x^2\,[x(1-x)\cdot]$ are given by $v_k(x) = w(x)\cdot u_k(x), k \geqslant 1$ where the weight-orthogonality measure $w(x)\,\mathrm{d}x$ is given by $w(x) = \frac{1}{x(1-x)}$, for the same eigenvalues, meaning

$$\mathcal{G}^*(v_k(x)) = \lambda_k v_k(x).$$

For instance, $v_1(x) = \frac{1}{1-x}$, $v_2(x) = 1$, $v_3(x) = 1-2x$, $v_4(x) = 1-5x+5x^2\ldots$.

Then, the following decomposition of the probability measure of $\mathcal{X}_{\tau=1}$ over the series of measures $v_k(x)\,\mathrm{d}x$ holds:

$$\mathbb{P}^{x_1}(X_\infty \in \mathrm{d}x) = \mathbb{P}^{x_1}(\mathcal{X}_1 \in \mathrm{d}x) = \sum_{k\geqslant 2} \frac{\mathrm{e}^{-\varepsilon k(k-1)/2}u_k(x_1)}{\langle v_k, u_k\rangle} v_k(x)\,\mathrm{d}x. \tag{13}$$

It coincides with the law of X_∞. When ε is small, this distribution approaches that (6) of P, which is beta $([\varepsilon^{-1}x_1]; \varepsilon^{-1} - [\varepsilon^{-1}x_1])$, arising in the discrete-time setting. At time $\tau = 1$, the law of the time-changed process of course keeps track of the initial condition x_1. But $\tau = 1$ is $t = \infty$ for the original process: the origin of the sensitivity to the initial conditions of the limit law for X_t can be clarified in this way.

3. The Pólya population gap process

In the Pólya expression of $p_{k,n}$, (1)–(2), we have $|2k-(N+n)| < N+n$ suggesting that what really drives the dynamics of N_n^1 is the gap process quantifying the excess population of type 1 balls within the urn, namely the quantity:

$$\Delta_n \equiv 2N_n^1 - (N+n) = N_n^1 - N_n^0 \text{ with initial condition } \Delta_0 = 2n_1 - N. \tag{14}$$

The support of the law of Δ_n is thus $\{\Delta_0 - n, \Delta_0 + n\}$. Given $\Delta_n = k$, we obtain the random walk model

$$\Delta_n = l \to l+1 \text{ with probability } p_{\frac{l+(N+n)}{2},n} = \frac{l+(N+n)}{2(N+n)} \tag{15}$$

$$\Delta_n = l \to l-1 \text{ with probability } q_{\frac{l+(N+n)}{2},n} = 1 - p_{\frac{l+(N+n)}{2},n}, \tag{16}$$

inhomogeneous in time. Is there a limit law on the random walk Δ_n? When is the random walk Δ_n transient in that: $\Delta_n \to \pm\infty$? To answer these questions, we shall consider a diffusion approximation of Δ_n.

3.1. Diffusion approximation of Δ_n

Let $\varepsilon = N^{-1} \to 0$. Suppose $n_1/N \to x_1 \in (0,1)$ $(N \to \infty)$. We shall consider

$$Y_t = \varepsilon \cdot \Delta_{[t/\varepsilon]}, \tag{17}$$

a scaled version of the process Δ_n and look for its dynamics, of diffusion type. Then, with $\delta Y_t = Y_{t+\varepsilon} - Y_t$ and $(\zeta_t; t \in \varepsilon\mathbb{Z})$ a standard iid Gaussian sequence, we wish to identify the drift f_t and variance $g^2_{\varepsilon,t}$ functions such that

$$\delta Y_t = f_t(Y_t)\varepsilon + \sqrt{\varepsilon} g_{\varepsilon,t}(Y_t)\zeta_{t+\varepsilon}.$$

Given $Y_t = y$, using the transition probabilities of Δ_n, we have from (15)–(16)

$$\mathbb{E}(\delta Y_t) = \varepsilon\mathbb{E}(\Delta_{[t/\varepsilon]+1} - \Delta_{[t/\varepsilon]}) = \varepsilon\left(2p_{\frac{y/\varepsilon+(N+t/\varepsilon)}{2},t/\varepsilon} - 1\right) = \varepsilon y/(1+t).$$

Thus, the drift is $f_t(y) = y/(1+t) \equiv ya_t$. Moreover, $\sigma^2(\delta Y_t) = \varepsilon^2$ in such a way that $g_{\varepsilon,t}(y) = \sqrt{\varepsilon}$. We conclude that Y_t obeys a diffusion equation in the sense of Itô with small additive noise:

$$\mathrm{d}Y_t = Y_t/(1+t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t, \qquad Y_0 = 2x_1 - 1. \tag{18}$$

3.2. Current distribution

Assume the initial composition N_0^1 of the urn is random. Let

$$\sigma_0^2 \equiv \sigma^2(Y_0) = 4\varepsilon^2\sigma^2\left(N_0^1\right)$$

be the variance of Y_0. If $Y_0 \overset{d}{\sim} \mathcal{N}\left(y_0, \sigma_0^2\right)$ is Gaussian or deterministic $Y_0 \overset{d}{\sim} \delta_{y_0}$, the law of Y_t keeps Gaussian in time with density:

$$p_t(y) = \frac{1}{\sqrt{2\pi}\sigma_t}\,\mathrm{e}^{-\frac{1}{2\sigma_t^2}(y-y_t)^2}, \qquad y \in \mathbb{R},$$

where $y_t = \mathbb{E}(Y_t)$ is the expected value of Y_t and $\sigma_t^2 \equiv v_t$ its variance.

With $a_t \equiv 1/(1+t)$, the dynamics of Y_t is linear of the form $\mathrm{d}Y_t = a_t Y_t\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t$, the drift being linear and inhomogeneous in time. If $\Psi_t(\lambda) = \log\mathbb{E}(\mathrm{e}^{\mathrm{i}\lambda Y_t})$ is the logarithm of the characteristic function of the law of Y_t, we have

$$\partial_t\Psi_t(\lambda) = a_t\lambda\partial_\lambda\Psi_t(\lambda) - \frac{\varepsilon}{2}\lambda^2$$

and $\Psi_t(\lambda) = \mathrm{i}\lambda y_t - \frac{\lambda^2}{2}v_t$. Identifying the real and imaginary parts of $\Psi_t(\lambda)$, we get the dynamics of the mean and variance of Y_t as

$$\dot{y}_t = a_t y_t, \qquad y_0 = 2x_1 - 1$$
$$\dot{v}_t = 2a_t v_t + \varepsilon, \qquad v_0 = \sigma_0^2.$$

Integrating, we find

$$y_t = y_0(1+t) \underset{t\text{ large}}{\sim} y_0 t \tag{19}$$

$$v_t = v_0(1+t)^2 + \varepsilon\int_0^t \left(\frac{1+t}{1+s}\right)^2 \mathrm{d}s \tag{20}$$

$$= v_0(1+t)^2 + \varepsilon(1+t)^2(1-(1+t)^{-1}) \sim (v_0+\varepsilon)t^2. \tag{21}$$

We also have (asymptotic normality)

$$\frac{Y_t - (2x_1 - 1)\,t}{\sqrt{v_0 + \varepsilon t}} \to \mathcal{N}(0, 1),$$

that is to say, recalling $Y_t = \varepsilon \cdot \Delta_{[t/\varepsilon]} = \frac{1}{N}\Delta_n N t = n$

$$\frac{\Delta_n - (2x_1 - 1)\,n}{\sqrt{v_0 + \varepsilon n}} \to \mathcal{N}(0, 1)$$

where the fluctuations of Δ_n turn out to be large (of the same order of magnitude in time as the mean).

3.3. Temporal correlations

Let $A_{s,t} \equiv \int_s^t a_\tau \,\mathrm{d}\tau = \log\left(\frac{1+t}{1+s}\right)$. The integral solution to the SDE (18) for Y_t is

$$Y_t = y_0 + \int_0^t \mathrm{e}^{A_{s,t}} (y_0 a_s \,\mathrm{d}s + \sqrt{\varepsilon}\,\mathrm{d}B_s).$$

The integral solution of $y_t = \mathbb{E}(Y_t)$ may be put under the form $y_t = y_0 + \int_0^t \mathrm{e}^{A_{s,t}} y_0 a_s \,\mathrm{d}s$. Thus, $\overline{Y}_t \equiv Y_t - y_t$ is a centered martingale process with

$$\overline{Y}_t = \sqrt{\varepsilon}\int_0^t \mathrm{e}^{A_{s,t}}\,\mathrm{d}B_s = \sqrt{\varepsilon}(1+t)\int_0^t (1+s)^{-1}\,\mathrm{d}B_s.$$

Let $\tau > 0$. It follows that

$$\begin{aligned}\mathrm{Cov}(Y_t, Y_{t+\tau}) \equiv \mathbb{E}(\overline{Y}_t\overline{Y}_{t+\tau}) &= \varepsilon(1+t)(1+t+\tau)\int_0^t (1+s)^{-2}\,\mathrm{d}s \\ &= \varepsilon t(1+t+\tau).\end{aligned} \tag{22}$$

This auto-covariance is positive and does depend on both $t, \tau > 0$ and not only on the time lag τ because Y_t is not a second order stationary Gaussian process, due to the time inhomogeneity of its drift.

4. A model with a variable degree of persistence: from Pólya to Friedman

We start with generalities on a model attributed to Friedman.

4.1. The Friedman model

Consider the (dual) discrete dynamics of Friedman defined as follows [8]:

$$N_{n+1}^1 = N_n^1 + \mathbf{1}_{U_{n+1} > P_n},$$

for which, given $N_n^1 = k \in \{n_1, \ldots, n_1 + n\}$

$$N_n^1 = k \to k+1 \text{ with probability } p_{k,n} = 1 - \frac{k}{N+n} = \frac{1}{2}\left(1 - \frac{2k - (N+n)}{N+n}\right) \tag{23}$$

$$N_n^1 = k \to k \text{ with probability } q_{k,n} = \frac{k}{N+n} = \frac{1}{2}\left(1 + \frac{2k - (N+n)}{N+n}\right). \tag{24}$$

The transition probabilities again depend on time n but have been exchanged as compared to the Pólya model (1)–(2): the Friedman model is dual to the Pólya model in the sense of Wall duality [3]. In this scheme, a ball is drawn at each time within the urn and its type is recorded; before proceeding to the subsequent drawing, replace the ball just drawn within the urn along with an additional ball of the *other* type (a stabilizing effect). At time n, we have $N + n$ balls as a whole among which N_n^1 are of type 1. This scheme is again that of a Markoff chain inhomogeneous in time. It was proposed by Friedman as a model of safety campaign. Quoting and rephrasing [5]: *Every time a ball of type 1 is drawn* (an accident occurs)*, the safety campaign is pushed harder* lowering the chance of a subsequent accident*, whereas if no accident occurs, the campaign slackens and the probability of an accident increases.* We may also switch to a different image. Suppose the types are the religious or political beliefs of a group of persons undergoing a propaganda campaign. It could be that in the process of convincing the unresolved people, these switch to the opposite camp instead of joining the ideas of the promoter. This adverse-campaign model is a Friedman urn process as well.

In the Friedman model, $P_n = N_n^1 \big/ (N + n) \to 1/2$ a.s. and the proportion of type 1 balls goes to a non-random limit traducing the stable (antipersistent) character of the Friedman urn process. The distribution of the Friedman's urn composition is ultimately concentrated on $1/2$, regardless of its initial composition.

4.2. From Pólya to Friedman: the PF urn model

This duality suggests to consider now the following urn dynamics: let $|\rho| \leqslant 1$ be a persistence coefficient determining the amplitude (or strength) of the reinforcement within the system. Consider the PF urn model defined by

$$N_n^1 = k \to k + 1 \text{ with probability } p_{k,n} = \frac{1}{2}\left(1 + \rho \frac{2k - (N + n)}{N + n}\right) \tag{25}$$

$$N_n^1 = k \to k \text{ with probability } q_{k,n} = \frac{1}{2}\left(1 - \rho \frac{2k - (N + n)}{N + n}\right). \tag{26}$$

Interpolating the Pólya ($\rho = 1$) and the Friedman ($\rho = -1$) urn models. When $\rho > 0$, a favorable situation at time n of gene 1 promotes the birth of a new gene of the same type and its advantage is enhanced or reinforced in future issues. In contrast, when $\rho < 0$, if in favorable position, gene 1 will be inhibited which is a stabilizing antipersistent feedback effect. A similar study involving simple transition probabilities as in (25)–(26) can be found in [9], although with no reference at all to the Pólya urn model and its relatives. Still, the relevance of the related concepts to real-world data was demonstrated convincingly enough. To be more precise, the authors in [9] discuss the applicability of this model to coarse-grained DNA sequences, written texts and financial data.

In the PF urn process, the proportion process $P_n = N_n^1 \big/ (N + n)$ is not a martingale. It satisfies

$$\mathbb{E}(P_{n+1} \mid P_n = x) = x + \frac{1 - \rho}{2(N + n + 1)}(1 - 2x).$$

4.3. The PF population gap process

Let $\Delta_n \equiv 2N_n^1 - (N + n)$ be the excess population process related to the PF process. We now have the random walk

$$\Delta_n = l \to l + 1 \text{ with probability } p_{\frac{l+(N+n)}{2},n} = \frac{1}{2}\left(1 + \rho \frac{l}{N + n}\right) \tag{27}$$

$$\Delta_n = l \to l - 1 \text{ with probability } q_{\frac{l+(N+n)}{2},n} = \frac{1}{2}\left(1 - \rho \frac{l}{N + n}\right). \tag{28}$$

When $\rho = 0$, Δ_n coincides with the usual fair random walk. As before, with $\varepsilon = N^{-1}$, we consider the scaling

$$Y_t = \varepsilon \cdot \Delta_{[t/\varepsilon]}.$$

Given $Y_t = y$, we now get

$$\mathbb{E}(\delta Y_t) = \varepsilon \mathbb{E}(\Delta_{[t/\varepsilon]+1} - \Delta_{[t/\varepsilon]}) = \varepsilon\big(2p_{\frac{y/\varepsilon+(N+t/\varepsilon)}{2},t/\varepsilon} - 1\big) = \varepsilon\rho\frac{y}{1+t}.$$

The drift is: $f_t(y) = \rho\frac{y}{1+t}$. Moreover, $\sigma^2(\delta Y_t) = \varepsilon^2$ in such a way that $g_{\varepsilon,t}(y) = \sqrt{\varepsilon}$ (small diffusion). Therefore Y_t obeys the SDE with linear drift and additive (small) Wiener noise:

$$\mathrm{d}Y_t = \rho Y_t/(1+t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t, \qquad Y_0 = 2x_1 - 1. \tag{29}$$

The solution for $y_t = \mathbb{E}(Y_t)$ now is

$$y_t = y_0(1+t)^\rho \underset{t \text{ large}}{\sim} y_0 t^\rho.$$

In antipersistent situations ($\rho < 0$), $y_t \to 0$ and the gap between type 0 and 1 balls concentrations shrink to 0: type 0 and 1 balls tend to equilibrate. In persistent situations ($\rho > 0$), $y_t \to \pm\infty$ depending on the sign of $2x_1 - 1$.

Starting from a Gaussian distribution, the law of Y_t remains Gaussian over time

$$p_t(y) = \frac{1}{\sqrt{2\pi}\sigma_t}\,\mathrm{e}^{-\frac{1}{2\sigma_t^2}(y-y_t)^2}, \qquad y \in \mathbb{R},$$

where $\sigma_t^2 \equiv v_t$ is the variance of Y_t. The dynamics of v_t is given by the Riccati equation

$$\dot{v}_t = 2\rho v_t/(1+t) + \varepsilon, \quad v_0,$$

leading if $\rho \neq 1/2$ to

$$v_t = v_0 + \int_0^t \left(\frac{1+t}{1+s}\right)^{2\rho}\left(\frac{2\rho}{1+s}v_0 + \varepsilon\right)\mathrm{d}s = v_0(1+t)^{2\rho} + \varepsilon\int_0^t \left(\frac{1+t}{1+s}\right)^{2\rho}\mathrm{d}s \tag{30}$$

$$= v_0(1+t)^{2\rho} + \frac{\varepsilon}{1-2\rho}(1+t)^{2\rho}((1+t)^{1-2\rho} - 1). \tag{31}$$

If $\rho = 1/2$, we are led to

$$v_t = v_0 + \int_0^t \left(\frac{1+t}{1+s}\right)\left(\frac{1}{1+s}v_0 + \varepsilon\right)\mathrm{d}s = v_0(1+t) + \varepsilon(1+t)\log(1+t).$$

To summarize the temporal behavior of the mean goes like $y_t \sim y_0 t^\rho$, whereas for the variance

$$v_t \sim \frac{\varepsilon}{1-2\rho}t \qquad \text{if} \quad \rho < 1/2 \tag{32}$$

$$v_t \sim \left(v_0 + \frac{\varepsilon}{2\rho - 1}\right)t^{2\rho} \qquad \text{if} \quad \rho > 1/2, \tag{33}$$

$$v_t \sim \varepsilon t\log(t) \qquad \text{if} \quad \rho = 1/2. \tag{34}$$

Equations (32)–(34) are not a novelty; they were first derived in ([9], equation (7)). With $\sigma_t \equiv v_t^{1/2}$, the following additional property is an immediate consequence of our computations:

$$\frac{Y_t - y_t}{\sigma_t} \overset{d}{\to} \mathcal{N}(0,1),$$

so that a central limit theorem holds.

The variance v_t undergoes a qualitative change at $\rho = 1/2$. Indeed, there are two variance regimes depending on the position of ρ with respect to $1/2$ corresponding to a critical regime. First, if $\rho \leqslant 1/2$ there is no influence of v_0 in the long run and the variance is of order ε. Second, $\rho < 1/2$ is a diffusive regime. Finally, if $\rho > 1/2$, we observe a super-diffusive regime.

In many cases, it can be useful to compare the process mean with its variance. For some point processes, the number of points variance within a time lag may grow typically like the number mean, as time gets large: if this property holds, the point process is called 'essentially Poissonian' as this is one of the Poisson characteristic feature. Anomalous (non-Poissonian) fluctuations for the random number of points may also arise. There are two cases: point processes can be 'super-homogeneous' if the number variance grows slower than the number mean and even 'hyper-uniform' (when variance growth saturates). This expresses some degree of regularity of the model under study. On the other hand, point processes can be 'sub-homogeneous' or critical (in the sense that the number variance grows faster than the number mean). Such point processes exhibit large fluctuations; see [2] for background. This last situation corresponds to our point process Δ_n because from (32), $v_t/y_t \to \infty$ whatever the value of ρ.

4.4. Temporal correlations

Assume $\rho \in [-1, 1]$ with $\rho \neq 1/2$. Let $A_{s,t} \equiv \int_s^t a_\tau \, d\tau = \log \left(\frac{1+t}{1+s}\right)^\rho$. The integral solution of the centered martingale process $\overline{Y}_t \equiv Y_t - y_t$ now is

$$\overline{Y}_t = \sqrt{\varepsilon} \int_0^t e^{A_{s,t}} \, dB_s = \sqrt{\varepsilon}(1+t)^\rho \int_0^t (1+s)^{-\rho} \, dB_s.$$

Let $\tau > 0$. Then

$$\mathrm{Cov}(Y_t, Y_{t+\tau}) = \mathbb{E}(\overline{Y}_t \overline{Y}_{t+\tau}) = \varepsilon(1+t)^\rho (1+t+\tau)^\rho \int_0^t (1+s)^{-2\rho} \, ds \tag{35}$$

$$= \frac{\varepsilon}{1-2\rho}(1+t+\tau)^\rho (1+t)^\rho [(1+t)^{1-2\rho} - 1]. \tag{36}$$

This covariance kernel do again depend on both $t, \tau > 0$ and not only on the time lag τ.

- When $\rho \in (1/2, 1]$, this covariance is a positive function of the variables $t, \tau > 0$ and so Y_t is positively self-correlated. For large $t \gg \tau$, we have $\mathrm{Cov}(Y_t, Y_{t+\tau}) \sim \frac{\varepsilon}{2\rho-1}(t+\tau)^\rho t^\rho$. For large $\tau \gg t$, we have $\mathrm{Cov}(Y_t, Y_{t+\tau}) \sim \frac{\varepsilon}{2\rho-1}(1+t)^\rho \tau^\rho$.
- When $\rho \in [-1, 1/2)$ this covariance still is a positive function of $t, \tau > 0$. Y_t is again positively correlated. For large t, we now have $\mathrm{Cov}(Y_t, Y_{t+\tau}) \sim \frac{\varepsilon}{1-2\rho}(t+\tau)^\rho t^{1-\rho}$ whereas for large $\tau \gg t$: $\mathrm{Cov}(Y_t, Y_{t+\tau}) \sim \frac{\varepsilon}{1-2\rho}(1+t)^{1-\rho}\tau^\rho$. In both cases, covariances are long-ranged and there is a qualitative change in the correlation structure at critical $\rho = 1/2$. The autocorrelation function requires normalizing. It is
$$\mathrm{Corr}(Y_t, Y_{t+\tau}) \equiv \frac{\mathrm{Cov}(Y_t, Y_{t+\tau})}{\sigma(Y_t)\sigma(Y_{t+\tau})}.$$
It can be computed using the expressions from (32) of the standard deviations $\sigma(Y_t) \equiv \sqrt{v_t}$ and $\sigma(Y_{t+\tau}) \equiv \sqrt{v_{t+\tau}}$.
- When $\rho = 0$, Y_t is Brownian motion $\sqrt{\varepsilon} B_t$ with well-known covariance $\varepsilon t \wedge (t+\tau) = \varepsilon t$.
- When $\rho = 1/2$ (critical case)
$$\mathrm{Cov}(Y_t, Y_{t+\tau}) = \varepsilon(1+t)^{1/2}(1+t+\tau)^{1/2} \log(1+t) \underset{t \text{ large}}{\sim} (t(t+\tau))^{1/2} \log t.$$
It exhibits a logarithmic dependence on t.

4.5. Diffusion approximations of the PF urn proportion model

In this section, we briefly address the problem of the diffusion approximations of the full PF urn proportion model, following the path used to get a Wright–Fisher approximation for the Pólya process.

Let $X_t = N^1_{[tN]}\big/(N + [tN]) = P_{[tN]}$. We still have

$$\delta X_t = \frac{\delta N^1_{[tN]} - X_t}{N + [tN] + 1}.$$

Given $X_t = x$, using the transition probabilities of the PF process, the law of $\delta N^1_{[tN]}$ now is

$$1 \text{ with probability } \frac{1-\rho}{2} + \rho x$$

$$0 \text{ with probability } \frac{1+\rho}{2} - \rho x.$$

Thus,

$$f_t(x)\varepsilon = \mathbb{E}(\delta X_t) = \frac{(1-x)}{N+[tN]+1}\left(\frac{1-\rho}{2} + \rho x\right) - \frac{x}{N+[tN]+1}\left(\frac{1+\rho}{2} - \rho x\right)$$

$$= \frac{1-\rho}{2}\frac{1-2x}{N+[tN]+1}$$

$$\varepsilon g^2_\varepsilon(x) = \sigma^2(\delta X_t) = \frac{(1-x)^2\left(\frac{1-\rho}{2} + \rho x\right) + x^2\left(\frac{1+\rho}{2} - \rho x\right)}{(N+[tN]+1)^2} - \mathbb{E}(\delta X_t)^2$$

$$= \frac{\frac{1-\rho}{2}\left[x^2 + (1-x)^2 - \frac{1-\rho}{2}(1-2x)^2\right] + \rho x(1-x)}{(N+[tN]+1)^2}.$$

As a result, defining the quadratic function $q_\rho(x) \equiv x^2 + (1-x)^2 - \frac{1-\rho}{2}(1-2x)^2$:

$$f_t(x) \sim \frac{1-\rho}{2}\frac{1-2x}{1+t} \qquad \text{and} \qquad g_{\varepsilon,t}(x) \sim \frac{\sqrt{\varepsilon}}{1+t}\sqrt{\frac{1-\rho}{2}q_\rho(x) + \rho x(1-x)}$$

are the new drift and local standard deviation entering in the full PF diffusion process. Note that both the drift and variance terms depend on t. Note also the symmetry property: $q_\rho(1-x) = q_\rho(x)$.

After the same time change (10) as that used for the Pólya process, we are led to a time-inhomogeneous diffusion on [0, 1] for $\mathcal{X}_\tau \equiv X_{t_\tau}$, $\tau \in [0, 1)$:

$$\mathrm{d}\mathcal{X}_\tau = \frac{1-\rho}{2}\frac{1-2\mathcal{X}_\tau}{1-\tau}\,\mathrm{d}\tau + \sqrt{\varepsilon}\sqrt{\frac{1-\rho}{2}q_\rho(\mathcal{X}_\tau) + \rho\mathcal{X}_\tau(1-\mathcal{X}_\tau)}\,\mathrm{d}B_\tau. \tag{37}$$

The new drift of the time-changed process is $\mathfrak{f}_\tau(x) = \frac{1-\rho}{2}(1-2x)/(1-\tau)$. When $\rho \neq 1$, this drift is positive when $x < 1/2$, negative when $x > 1/2$. Therefore, 1/2 is a stable point of the noiseless dynamics ($\varepsilon = 0$), say $\mathcal{X}^*_\tau$, whose integral solution can easily be obtained under the form

$$\mathcal{X}^*_\tau = \tfrac{1}{2}[1 - (1-2\mathcal{X}^*_0)(1-\tau)^{1-\rho}] \underset{\tau \nearrow 1}{\rightarrow} \tfrac{1}{2}. \tag{38}$$

Equation (37) should be regarded as a small diffusive perturbation of equation (38).

Up to the coefficient ε, the volatility coefficient of the time-changed process in (37) is

$$\mathfrak{g}^2(x) \equiv \frac{1-\rho}{2}q_\rho(x) + \rho x(1-x).$$

The quadratic function $x \to \mathfrak{g}^2(x)$ is always symmetric with respect to $1/2$ with $\mathfrak{g}^2(1/2) = 1/4$, $\mathfrak{g}^2(0) = \mathfrak{g}^2(1) = \frac{1-\rho^2}{4} \geqslant 0$. When $\rho \neq 0$, with $x \in [0,1]$, we easily obtain the factorized expression

$$\mathfrak{g}^2(x) = \rho^2 (x - x_-)(x_+ - x) \qquad \text{where} \quad x_\pm = \frac{1}{2}\left(1 \pm \frac{1}{|\rho|}\right), \tag{39}$$

with $x_- \leqslant 0$ and $x_+ \geqslant 1$. Note that the function $\rho \to \mathfrak{g}^2(x)$ is invariant under the transformation $\rho \to -\rho$.

In sharp contrast to the much studied Wright–Fisher diffusions ($\rho = 1$), we are not aware of any working paper considering such generalized diffusion processes as in (37).

Special cases include:

- If $\rho = 1/2$
$$\mathfrak{f}_\tau(x) = \frac{1}{4}\frac{1-2x}{1-\tau} \qquad \text{and} \qquad \mathfrak{g}^2(x) = \frac{1}{4}\left(x + \frac{1}{2}\right)\left(\frac{3}{2} - x\right).$$
- If $\rho = -1/2$
$$\mathfrak{f}_\tau(x) = \frac{3}{4}\frac{1-2x}{1-\tau} \qquad \text{and} \qquad \mathfrak{g}^2(x) = \frac{1}{4}\left(x + \frac{1}{2}\right)\left(\frac{3}{2} - x\right).$$
- If $\rho = -1$ (Friedman urn), $\mathfrak{f}_\tau(x) = \frac{1-2x}{1-\tau}$ and $\mathfrak{g}^2(x) = x(1-x)$. We have to consider the Friedman diffusion which is dual to that of Wright–Fisher, namely,
$$\mathrm{d}\mathcal{X}_\tau = \frac{1 - 2\mathcal{X}_\tau}{1-\tau}\,\mathrm{d}\tau + \sqrt{\varepsilon}\sqrt{\mathcal{X}_\tau(1-\mathcal{X}_\tau)}\,\mathrm{d}B_\tau.$$
- If $\rho = 0$, $\mathfrak{g}^2(x) = 1/4$ and the diffusion reduces to an Ornstein–Uhlenbeck like one
$$\mathrm{d}\mathcal{X}_\tau = \frac{1}{2}\frac{1-2\mathcal{X}_\tau}{1-\tau}\,\mathrm{d}\tau + \sqrt{\frac{\varepsilon}{4}}\,\mathrm{d}B_\tau.$$

5. Conclusion

We investigated a parametric urn process called the Pólya–Friedman urn model, involving a persistence effect. The study includes the diffusion approximations of both the discrete Pólya–Friedman proportion process and the population gap random walk. We have shown that the structure of the former is a generalized time-inhomogeneous Wright–Fisher diffusion. The structure of the latter is a stochastic differential equation involving a linear time-inhomogeneous drift with an additive (small) Wiener term; the correlation structure of the latter was shown to present a 'dramatic' change at a critical value of the persistence parameter.

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